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A COUNTEREXAMPLE IN COPOSITIVE APPROXIMATION

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ABSTRACT

The present paper gives a converse result by showing that there exists a function $f \in C_{[-1,1]}$, which satisfies that $\operatorname{sgn}(x)f(x) \geq 0$ for $x \in [-1,1]$, such that

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f,1)}{E_n(f)} = +\infty,$$

where $E_n^{(0)}(f,1)$ is the best approximation of degree n to f by polynomials which are copositive with it, that is, polynomials P with $P(x)f(x) \geq 0$ for all $x \in [-1,1]$, $E_n(f)$ is the ordinary best polynomial approximation of f of degree n.

1. Introduction

Denote by $C^k_{[-1,1]}$ the class of real functions $f(x)$, which have k continuous derivatives on the interval $[-1,1]$, with norm $\|f\| = \max\{|f(x)| : -1 \leq x \leq 1\}$, $C_{[-1,1]} = C^0_{[-1,1]}$, $\Delta^0(r)$ the class of real functions which alternate sign r times on $[-1,1]$. Let Π_n the class of algebraic polynomials of degree at most n.

As usual, let $E_n(f)$ denote the best approximation to $f \in C_{[-1,1]}$ by algebraic polynomials of degree n, $E_n^{(0)}(f,r)$ the best approximation of $f \in C_{[-1,1]} \bigcap \Delta^0(r)$ by polynomials of degree n which are copositive with it, that is, polynomials $P(x) \in \Pi_n$ with $f(x)P(x) \geq 0$ for all $x \in [-1,1]$. Let

$$\omega_k(f,t) = \max\left\{\|\Delta_h^k f(x)\| : \ x, x+h \in [-1,1], \ 0 < h \leq t\right\},$$

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where

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh).$$

On the topic of copositive approximation, works [1], [3], [4] investigated the Jackson type theorems. The latest result is that for $f \in C_{[-1,1]} \bigcap \Delta^0(r)$,

$$E_n^{(0)}(f,r) \le C_r \omega(f, n^{-1}),$$

where C_r is a constant depending only upon r, which is due to D. Leviatan [1].

However, unlike in the general comonotone approximation, in which many counterexamples have been established showing that the degree of comonotone approximation is much worse than that of ordinary best approximation, in copositive case, we cannot find much information on the relation between $E_n^{(0)}(f,r)$ and $E_n(f)$. A very clear fact is that if $E_n^{(0)}(f,r)$ is not worse than $E_n(f)$, that is, $E_n^{(0)}(f,r) = O(E_n(f))$, then we do not need to discuss the Jackson type inequalities anymore, since complete results have been established for $E_n(f)$. For example, if $r = 0$, for all nonnegative functions $f \in C_{[-1,1]}$, it is obvious that

$$E_n^{(0)}(f,0) \le 2E_n(f),$$

we do not need to go further.

For $r \ge 1$, however, a very likely result is that one can find a function $f \in C_{[-1,1]} \bigcap \Delta^0(r)$ such that

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f,r)}{E_n(f)} = +\infty.$$

In fact it is enough to show* that there is a function $f \in C_{[-1,1]} \bigcap \Delta^0(1)$ such that

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f,1)}{E_n(f)} = +\infty.$$

This is a conjecture we raised in [6], in which a counterexample in L^p space for $2 < p < \infty$ for $r = 0$ is given.

The present paper will prove a converse result in copositive approximation, which in particular gives an affirmative answer to the above conjecture.

* We will give more information about this fact later in Section 3.

2. Result and Proof

THEOREM 1: *There is a function* $f \in C^1_{[-1,1]} \bigcap \Delta^0(1)$ *such that*

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f,1)}{E_n(f)} = +\infty.$$

Indeed, Theorem 1 can be deduced from the following stronger statement by applying Jackson Theorem.

THEOREM 2: *There is a function* $f \in C^1_{[-1,1]} \bigcap \Delta^0(1)$ *such that*

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f,1)}{\omega_4(f,n^{-1})} = +\infty.$$

LEMMA: *Suppose that* $a > 0$, $\alpha(x) = \frac{x^2}{x^2-a^2}$, $g(x,a) = xe^{\alpha(x)}$, $x \in (-a,a)$, *then*

$$|g'(x,a) - 1| \le Ca^{-2}x^2, \quad |x| < a,$$

and

$$|g''(x,a)| = O(a^{-2}x), \quad |x| < a,$$

where here and in the sequel, C always denotes an absolute constant which may vary from one occurrence to another even in the same line.

Proof: The idea for proof is the same as that in Lemma 1 of [5]. ∎

Proof of Theorem 2: Let $C^\infty_{[-1,1]}$ be the class of real functions, infinitely differentiable on $[-1,1]$, and let

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

We begin with construction of a sequence of functions, $f_n(x)$, satisfying

$$f_n(x) \in C^\infty_{[-1,1]}, \tag{1}$$

$$\operatorname{sgn}(x) f_n(x) \ge 0, \tag{2}$$

$$\left\| f_n(x) - x^3 + n^{5/2}x \right\| \approx n^{-29/8}, \tag{3}$$

for $m = 0, 1$,

$$\|f_n^{(m)}(x)\| = O(1), \tag{4}$$

$$f_n^{(m)}(0) = 0, \tag{5}$$

and, for sufficiently large n,

$$\operatorname{sgn}(x) f_n''(x) \geq 0, \tag{6}$$

where $A_n \approx B_n$ means that there is a positive constant M independent of n such that

$$M^{-1} A_n \leq B_n \leq M A_n.$$

In fact, if we let

$$\bar{g}_n(x) = n^{-5/2} g(x, n^{-9/8}) + x^3 - n^{-5/2} x, \quad x \in (-n^{-9/8}, n^{-9/8}),$$

then

$$\bar{g}_n'(x) = n^{-5/2} \left(g'(x, n^{-9/8}) - 1 \right) + 3x^2.$$

By the Lemma,

$$\bar{g}_n'(x) \leq 3x^2 + O(n^{-1/4} x^2)$$

and

$$\bar{g}_n'(x) \geq 3x^2 - O(n^{-1/4} x^2)$$

for $x \in (-n^{-9/8}, n^{-9/8})$. Hence

$$\bar{g}_n'(0) = 0,$$

and for sufficiently large n,

$$\bar{g}_n'(x) \geq 0$$

for $x \in (-n^{-9/8}, n^{-9/8})$. Combining this with the fact $\bar{g}_n(0) = 0$, we get

$$\operatorname{sgn}(x) \bar{g}_n(x) \geq 0$$

for $x \in (-n^{-9/8}, n^{-9/8})$. By the Lemma again,

$$\bar{g}_n''(x) \geq 6x - O(n^{-1/4} |x|)$$

for $x \in [0, n^{-9/8})$, or

$$\bar{g}_n''(x) \le 6x + O(n^{-1/4}|x|)$$

for $x \in (-n^{-9/8}, 0]$. Thus for sufficiently large n,

$$\operatorname{sgn}(x)\bar{g}_n''(x) \ge 0$$

for $x \in (-n^{-9/8}, n^{-9/8})$. Put

$$f_n(x) = \begin{cases} x^3 - n^{-5/2}x, & |x| \ge n^{-9/8}, \\ \bar{g}_n(x), & |x| < n^{-9/8}. \end{cases}$$

From the above discussion and direct calculations, we can verify (1)–(6).

Because of (6), applying the result of copositive approximation from [1], we can find a polynomial,

$$q_n(x) = \sum_{j=0}^{N_n^*-2} a_j x^j \in \Pi^*_{N_n^*-2},$$

such that

(7) $$\|f_n''(x) - q_n(x)\| < n^{-4},$$

where N_n^* is a natural number depending on n, and Π_n^* is the class of polynomials $P(x)$ of degree n satisfying

$$\operatorname{sgn}(x)P(x) \ge 0$$

for $x \in [-1, 1]$. Write

$$Q_n(x) = \int_0^x dx_1 \int_0^{x_1} q_n(x_2)dx_2.$$

Clearly $Q_n \in \Pi^*_{N_n^*}$ and, by (5) and (7),

$$\|f_n - Q_n\| \le \|f_n'' - q_n\| < n^{-4}.$$

Evidently,

(8) $$Q_n^{(j)}(0) = 0, \quad j = 0, 1,$$

which, together with (3) and (7), gives

(9) $$\left\|Q_n(x) - x^3 + n^{-5/2}x\right\| \approx n^{-29/8},$$

and

$$\|Q_n^{(m)}\| = O(1) \tag{10}$$

for $m = 0, 1$.

Let

$$F_l(x) = \sum_{j=1}^{l} n_j^{-1/4} Q_{n_j}(x),$$

$$P_l(x) = F_{l-1}(x) + n_l^{-1/4}(x^3 - n_l^{-5/2}x),$$

where $\{n_l\}$ is a subsequence of natural numbers chosen by induction: Choose large enough n_1 satisfying (6),

$$n_{l+1} = 2\left[n_l^{16} + N_{n_l}^* + \|F_l^{(4)}\|^8 + 1\right] \tag{11}$$

for $l = 1, 2, \cdots$, where $[x]$ is the greatest integer not exceeding x.

It is not difficult to see that

$$P_l'(0) = -n_l^{-11/4} \tag{12}$$

by (8), and

$$\|F_l - P_l\| = n_l^{-1/4}\|Q_{n_l} - x^3 + n_l^{-5/2}x\| \approx n_l^{-31/8} \tag{13}$$

by (9). Now (12) and (13) imply that

$$n_l^{9/8}\|F_l - P_l\| \le C|P_l'(0)|. \tag{14}$$

Let $r(x)$ be any polynomial in $\Pi_{n_l}^*$ (in this case $r'(0) \ge 0$). Then (12), together with Bernstein's inequality (see [2]) gives us

$$|P_l'(0)| \le |P_l'(0) - r'(0)| \le Cn_l\|P_l - r\| \le Cn_l(\|P_l - F_l\| + \|F_l - r\|). \tag{15}$$

Combining (13), (14) and (15), for l large enough, we get

$$\|F_l - r\| \ge Cn_l^{1/8}\|F_l - P_l\| \ge Cn_l^{-15/4}. \tag{16}$$

Define

$$f(x) = \sum_{j=1}^{\infty} n_j^{-1/4} Q_{n_j}(x).$$

It is clear that $f \in C^1_{[-1,1]}$, $\text{sgn}(x)f(x) \geq 0$ for $x \in [-1,1]$ by (10) and the fact $Q_n(x) \in \Pi^*_{N^*_n}$. For any $r \in \Pi^*_{n_l}$, in view of (10),

$$\|f(x) - r(x)\| \geq \|F_l - r\| - O\left(\sum_{j=l+1}^{\infty} n_j^{-1/4}\right).$$

Applying (11) and (16) we have

$$\|f(x) - r(x)\| \geq C(n_l^{-15/4} - O(n_l^{-4})) \geq Cn_l^{-15/4},$$

so that

$$E^{(0)}_{n_l}(f,1) \geq Cn_l^{-15/4}. \tag{17}$$

At the same time, due to (9) and (11),

$$\omega_4(f, n_l^{-1}) \leq \|F^{(4)}_{l-1}\|n_l^{-4} + n_l^{-1/4}\omega_4\left(Q_{n_l}(x) - x^3 + n_l^{-5/2}x, n_l^{-1}\right) + O\Big(\sum_{j=l+1}^{\infty} n_j^{-1/4}\Big)$$

$$= O(n_l^{-31/8}) + O(n_l^{-31/8}) + O(n_l^{-4}). \tag{18}$$

Combining (17) and (18) for sufficiently large l yields that

$$\frac{E^{(0)}_{n_l}(f,1)}{\omega_4(f,n_l^{-1})} \geq Cn_l^{1/8},$$

and the proof of Theorem 2 is completed. ∎

3. Remark

Let $T_1(f,x) := f(x) \in C^1_{[-1,1]} \bigcap \Delta^0(1)$ be the function we established in Theorem 2. Let

$$S(T_1, x) = T_1(f, 2x-1), \quad x \in [0,1],$$

$$T_2(f,x) = S(T_1, x^2), \quad x \in [-1,1];$$

evidently, $T_2(f) \in C_{[-1,1]} \bigcap \Delta^0(2)$. In a similar way, we have a function $T_4(f,x) := T_2(T_2(f),x) \in C_{[-1,1]} \bigcap \Delta^0(4)$. Now let

$$T_3(f,x) = T_4(f, 3x/4 + 1/4),$$

then $T_3(f) \in C_{[-1,1]} \bigcap \Delta^0(3)$. The same discussions are applied to $Q_{n_j}(x)$ and $P_l(x)$, too. Note that now $T_j(P_l, x)$, $j = 1,2,3,4$, have degree at most $4N^*_{n_{l-1}}$ (in the present cases, we also need to change slightly the definition of $\{n_l\}$). Then by the same method in the proof of Theorem 2, we can obtain that for $r \in \Pi_{4n_l} \bigcap \Delta^0(j)$,

$$\|T_j(F_l) - r\| \geq \max_{1-2/j \leq x \leq 1} |T_j(F_l, x) - r(x)| \geq C n_l^{-15/4}, \quad j = 1,2,3,4.$$

At the same time let $p^*(x)$ be the polynomial of best approximation of degree n_l to $f(x)$. Evidently,

$$E_{4n_l}(T_j(f)) \leq \|T_j(f) - T_j(p^*)\| = O(n_l^{-31/8}), \quad j = 1,2,3,4.$$

Thus we have functions $T_j(f) \in C_{[-1,1]} \bigcap \Delta^0(j)$, $j = 1,2,3,4$, such that

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(T_j(f), j)}{E_n(T_j(f))} = +\infty.$$

The general case of a function $T_r(f) \in C_{[-1,1]} \bigcap \Delta^0(r)$ for which

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(T_r(f), r)}{E_n(T_r(f))} = +\infty$$

can be treated in a similar way.

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